Int J Multiphase Flow Vol 18, No 1, pp 65–87, 1992
Printed in Great Britain

0301-9322/92 $5.00 + 0.00

MOMENTUM INTERACTIONS IN DISPERSED FLOW: AN AVERAGING AND A VARIATIONAL APPROACH

C. PAUCHON[1]† and P. SMEREKA[2]‡
[1]SINTEF Multiphase Flow Laboratory, 7034 Trondheim, Norway
[2]Courant Institute, 251 Mercer St, New York, NY 10012, U.S.A.

(Received 15 June 1990; in revised form 8 May 1991)

Abstract—It is shown that spatial averaging of the local instantaneous conservation laws for inviscid incompressible dispersed flow yields identical results to equations obtained from a variational principle, in the dilute limit. The equations deduced from the variational approach extend easily to the nondilute case. The drift flux concept reduces the 4×4 system of equations to a 2×2 system which is cast into the framework of continuous Hamiltonian systems. The hyperbolicity of the equations is examined and it is concluded that the system is ill-posed for isotropic dispersions. It is then demonstrated that the model is Lyapunov stable, provided it is hyperbolic.

Key Words: bubbly flow, averaging, variational methods

1. INTRODUCTION

1.1. Background

The approach to two-phase flow modelling that is now widely used in computer codes such as TRAC (see *TRAC PD2 Manual* 1982), RELAP 5 (Ransom *et al.* 1981). CATHARE (Micaelli 1987) or OLGA (Bendiksen *et al.* 1988) is based on averaging of the original local instantaneous conservation equations for mass, momentum and energy. Averaging may be done in time, space, over an ensemble or in some combination of these. Details may be found in Panton (1978), Vernier & Delhaye (1968), Bouré *et al.* (1975), Ishii (1975), Yadigaroglu & Lahey (1976), Agee *et al.* (1978), Lyckowski (1978), Nigmatulin (1978, 1979), Banerjee & Chan (1980) and Drew (1983) amongst others.

Averaging makes the mathematical aspect of the model more tractable, but in the process, information regarding local gradients is lost. Consequently, closure relations in terms of the averaged variables have to be supplied in order to model the local mass, momentum and energy transfer terms, both at the wall and at the interface. In the following, we are concerned with the derivation of these closure laws, or constitutive relations, in the case of dispersed incompressible two-phase flow.

Incompressible dispersed flows constitute an interesting case study because they can be considered as an approximation to a number of two-phase flow situations, yet the corresponding phasic interaction terms are relatively well defined. In this case we concentrate on the potential flow approximation of dispersed flow, we show in particular that the complete set of constitutive relations reduces to: the virtual mass acceleration; the continuous phase velocity perturbation term, loosely called "Reynolds stress" here; and the difference between phasic and interfacial pressure on the continuous phase side.

It turns out that these three momentum transfer terms provide a complete picture of energy conserving momentum interaction terms, occurring in the specific situation of incompressible dispersed flows. Moreover, the analysis shows that these three terms have a common physical basis; the kinetic energy associated to the continuous phase velocity perturbation.

While this conclusion is not very surprising, it turns out that these three interaction effects are generally studied rather independently from one another. Drew *et al.* (1979) have analyzed the virtual mass force from the point of view of material frame indifference. Voinov (1973) derived the

†Present address: Institut Francais du Petrole, BP 311, 92506 Rueil Malmaison, France.
‡Present address: Department of Mathematics, University of California, Los Angeles, CA 90024-1555, U.S.A.

virtual mass force applied to a single sphere with infinite dilution. Biesheuvel & van Wijngaarden (1984) pointed out the importance of the Reynolds stress type terms or velocity distribution terms, and Pauchon & Banerjee (1986) showed that accounting for the phasic pressure difference leads to a model having real characteristics up to a critical void fraction, which was conjectured to signal the transition from the bubbly to slug flow regime.

Wallis (1989) questioned the applicability of these models by performing simple tests on the various models. One of his conclusions was that by approaching each constitutive relation separately, some effects might be accounted for twice in the model formulation. The previously mentioned conclusion that the three momentum exchange terms have the same physical basis would seem to validate Wallis's point. However, we show by a variational approach that these three terms give a complete picture of the momentum interaction effects in the case of inviscid incompressible flow.

Variational principles provide a powerful alternative for the derivation of the constitutive relations. Instead of averaging the local instantaneous equations of motion, we start from a macroscopic energy functional written in terms of the averaged variables. The equations of motion are then derived systematically from this energy functional. The traditional problem of deriving the macroscopic forces in terms of averaged variables is eluded, as long as the energy functional itself can be written in terms of the averaged variables. The remaining problem is to classify the terms appearing in the resulting equations of motion, in order to derive the constitutive relations which were identified but unresolved by the averaging approach.

Thus, it is seen that the variational and averaging approach are complementary. The variational approach provides the analytical equations of motion explicitly, but in the process the physical significance of each of the individual terms is lost. The averaging approach does exactly the opposite: the physical meaning of each constitutive relation is generally understood, but their analytical form remains *a priori* unknown. The Lagrangian variational approach was used previously in the derivation of the equations of motion for dispersed mixtures by Geurst (1985a, b, 1986). Geurst presented conclusive results on the form of the virtual mass acceleration, he showed that the resulting acceleration is objective.

The Hamiltonian formalism and Lyapunov stability of multiphase flows has been studied by Holm & Kupershmidt (1986a, b). Their analysis focuses on the local equations. While it is well known that a single pressure model is ill-posed, they show that the introduction of surface tension can provide Lyapunov stability. Furthermore, they show that the use of multiple pressures and the introduction of interfacial terms can provide Lyapunov stability and hyperbolicity. We shall show that similar results can be achieved from averaged equations where the quantities associated with the interfacial dynamics appear as closure relations of the averaged two-phase model.

1.2. Summary

This study is concerned with the propagation of void or concentration waves that occur in dispersed flows, e.g. gas bubbles in a liquid. In particular, we investigate the momentum interactions that occur when the two phases move relative to each other. To clarify these interactions, we will ignore the compressibility of both the gas and liquid phase, which are important in studying pressure wave propagation. Viscous forces and mass transfer between phases is also ignored in this study. We shall derive effective equations of motion for this mixture using both an averaging approach and variational principles. The outcome of the following study may be summarized as follows:

- It has been demonstrated that, in the dilute limit, the equations of motion for a bubbly liquid obtained by volume averaging the local instantaneous conservation laws are the same as those deduced from a variational principle. The variational approach provides the equations of motion in the nondilute case.
- It is shown that a correction term must be added to the virtual mass acceleration term for a single bubble given by Voinov (1973). The correction term accounts for the flux of bubbles crossing the averaging volume. The variational principle gives the form of this term for the nondilute case.

- The variational approach reveals that the Reynolds stress, the phasic pressure difference and the virtual mass acceleration are manifestations of the contribution by the relative motion to the overall kinetic energy of the system.
- It is shown that the 4×4 system deduced using the two approaches simplifies to a 2×2 system in terms of the drift flux and void fraction. The Hamiltonian structure of these equations is formulated and the relationship between Lyapunov stability and hyperbolicity is examined. It is found that the hyperbolicity is entirely determined by the form of the virtual mass.

In more detail, section 2 begins with the volume-averaged instantaneous mass and momentum conservation equations. In the inviscid incompressible limit, the constitutive relations are identified. These relations are derived in the dilute limit by considering the motion of a single bubble in an unbounded liquid.

Section 3 outlines the derivation of the equations of motion from a variational principle. The momentum conservation equations are written down with a single unknown coefficient $m(\epsilon_G)$ of the kinetic energy associated with the velocity perturbations due to the presence of the dispersed phase. It is then shown that in the dilute limit these equations reduce to the equations obtained from averaging.

The reduction of the system by use of the drift flux (Zuber 1964) is discussed in section 4. The resulting system is a 2×2 system in conservation law form. The Hamiltonian formalism for the reduced system is explicit. It is shown that the hyperbolicity condition reduces to the Lyapunov stability condition, and that this condition depends only on the virtual mass coefficient. In particular, the results imply that the equations of motion for isotropic dispersed flows are ill-posed.

2. THE VOLUME-AVERAGED MODEL

In this section we present the volume-averaged conservation laws for mass and momentum and the constitutive relations needed for closure. In the dilute limit of dispersed flow, we will derive the constitutive relations for the virtual mass acceleration, the velocity perturbation term and the phasic pressure difference term.

2.1. Averaged conservation equations

For a detailed account of the volume-averaging procedure, the reader is referred to Vernier & Delhaye (1968), Ishii (1975) and Banerjee & Chan (1980). Here we start our analysis from the mass and momentum conservation equations of field k, as defined by Banerjee & Chan (1980):

$$\frac{\partial \epsilon_k}{\partial t} + \frac{\partial \epsilon_k \langle u_k \rangle}{\partial z} = 0 \qquad [1a]$$

and

$$\rho_k \left\{ \frac{\partial}{\partial t} \epsilon_k \langle u_k \rangle + \frac{\partial}{\partial t} \left[\epsilon_k (\langle u_k \rangle^2 + \langle u_k'^2 \rangle) \right] \right\} + \epsilon_k \frac{\partial \langle p_k \rangle}{\partial z} - \frac{\partial}{\partial z} \epsilon_k \langle \mathbf{n}_z \cdot (\tau_k \cdot \mathbf{n}_z) \rangle - \Delta p_{ki} \frac{\partial \epsilon_k}{\partial z}$$
$$= \epsilon_k \rho_k \langle \mathbf{F}_k \cdot \mathbf{n}_z \rangle - \langle \mathbf{n}_k \cdot \mathbf{n}_z \, \Delta p_{ki}' \rangle_i + \langle \mathbf{n}_z \cdot (\tau_k \cdot \mathbf{n}_k) \rangle_i + \langle \mathbf{n}_z \cdot (\tau_k \cdot \mathbf{n}_{kw}) \rangle_w, \qquad [1b]$$

where

$$\langle f_k \rangle = \frac{1}{V_k} \int_{V_k} f_k \, dV \quad \text{and} \quad \langle f_k \rangle_i = \frac{1}{V} \int_{a_i} f_k \, dS.$$

Equation [1a] is the averaged mass conservation equation and [1b] expresses the conservation of momentum in the z-direction. Here ϵ_k is the volume fraction of field k in the volume V, $\mathbf{n}_k$ is the outward drawn normal vector on the surface of field, $\mathbf{n}_z$ is the unit vector in the z-direction, a_i represents the interfacial area of field k and a_{kw} is the area of contact between field k and the wall (see also figure 1). The other variables are: ρ_k, the density; u_k, the local velocity in the z-direction; p_k, the pressure; τ_k, the shear stress tensor; and $\mathbf{F}_k$, the body force. In analogy with bubbly flow systems, in the following we will use the subscript G for the dispersed phase or gas phase, and L for the continuous phase or liquid phase. The $\langle u_k'^2 \rangle$ term is loosely called the

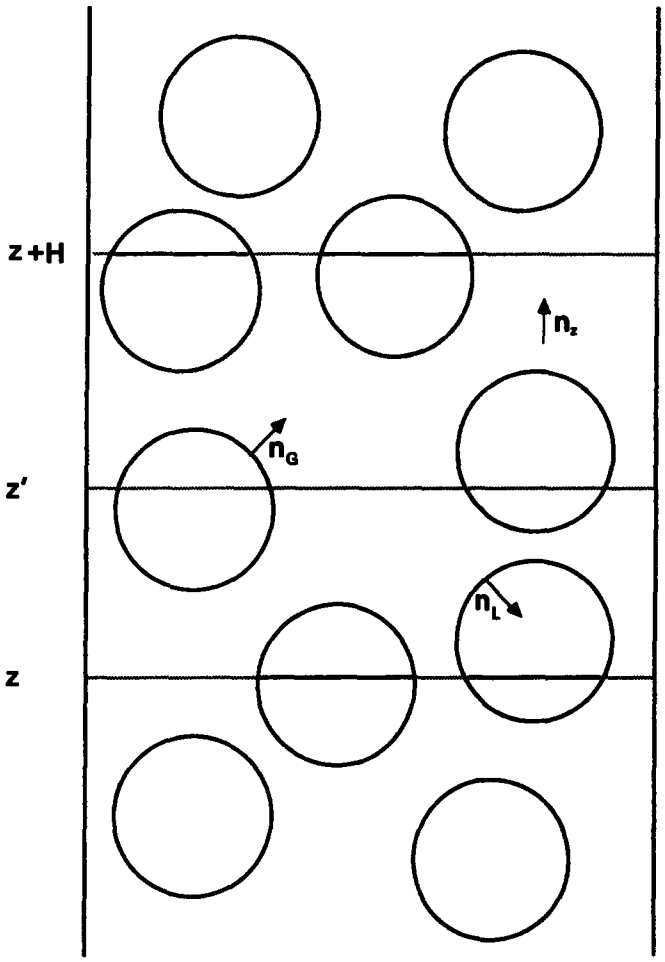

Figure 1. A schematic drawing of the bubbly flow and the averaging volume. The averaging volume runs from z to $z + H$. The solid lines inside the bubbles that cross the averaging volume represent $A_G(z)$ and $A_G(z + H)$. The outward drawn normals, $\mathbf{n}_G$ and $\mathbf{n}_L$ are shown only once each but are understood to be given for all surfaces. The z' slice is the coordinate used within the averaging volume. The mean flow is in the $\mathbf{n}_z$-direction

Reynolds stress and will be discussed below. The Δp_{ki} and $\Delta p'_{ki}$ terms were introduced by Banerjee & Chan (1981) and are defined as:

$$\Delta p_{ki} = \frac{V}{a_i} \langle p_{ki} \rangle_i - \langle p_k \rangle \qquad [2]$$

and

$$\Delta p'_{ki} = p_{ki} - \frac{V}{a_i} \langle p_{ki} \rangle_i. \qquad [3]$$

Notice that [2] and [3] imply that the interfacial pressure has been decomposed as follows:

$$p_{ki} = \langle p_k \rangle + \Delta p_{ki} + \Delta p'_{ki}. \qquad [4]$$

This decomposition has the advantage of distinguishing the separate contributions in the momentum exchange terms as discussed below. Wallis (1990) presents a different approach. He focuses on the tensorial nature of the interfacial pressure. This aspect of the interfacial pressure must also be present in [1b]; however, we are only concerned with the net force produced by such a tensor and therefore do not examine this aspect of the interfacial pressure.

It is important to realize that [1b] is an exact spatial average of the Navier–Stokes equation for a mixture of two phases. Accounting for material crossing the averaging volume has been given proper consideration. The only approximation comes when trying to deduce constitutive relations to close the system. For a detailed derivation the reader is referred to Delhaye & Achard (1976).

Let us discuss briefly the physical relevance of some of the terms in [1b]. In the steady-state creeping flow approximation, the term $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_i$ leads to the form drag. In the unsteady creeping flow approximation, this terms contains also the normal component of the Basset force, while the friction drag and the tangential component of the Basset force are included in the term $\langle \mathbf{n}_z \cdot (\tau_k \cdot \mathbf{n}_k) \rangle_i$ (e.g. Basset 1961). In the inviscid potential flow approximation, the interfacial pressure integral leads also to the virtual mass and virtual buoyancy force.

Together with the momentum transfer term associated to the term $\Delta p_{ki} \partial \epsilon_k / \partial z$, the interfacial pressure term $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_i$ and the viscous stress integrals $\langle \mathbf{n}_z \cdot (\tau_k \cdot \mathbf{n}_k) \rangle_i$ constitute the set of terms representing momentum exchange between the phases.

The $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_i$ term is the "symmetric" virtual mass acceleration. By "symmetric" we mean that it occurs in both the dispersed and continuous phase momentum equations, with opposite signs. This term represents the momentum exchange between the phases and it must be symmetric if $p_{Li} = p_{Gi}$, which is clear if it is borne in mind that there is no net force associated with this term. This idea is made more precise by the use of jump conditions (Banerjee & Chan 1980).

In the inviscid incompressible approximation the momentum equation simplifies to

$$\rho_k \left[\frac{\partial}{\partial t} \epsilon_k \langle u_k \rangle + \frac{\partial}{\partial z} \left(\epsilon_k \langle u_k \rangle^2 + \epsilon_k \langle u_k'^2 \rangle \right) \right] + \epsilon_k \frac{\partial \langle p_k \rangle}{\partial z} - \Delta p_{ki} \frac{\partial \epsilon_k}{\partial z} = - \langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_i. \quad [5]$$

The $\langle u_k'^2 \rangle$ term represents the Reynolds stress term. Here we will consider only the Reynolds stress in the continuous phase which is due to the presence of the dispersed phase. In general, we have

$$\mathbf{u}_k = \langle \mathbf{u}_k \rangle + \mathbf{u}'_k; \quad [6]$$

we will take the mean flow to be in the z-direction, so we have $\langle \mathbf{u}_k \rangle = \langle u_k \rangle \mathbf{n}_z$ and it then follows that

$$\langle \mathbf{u}_k \cdot \mathbf{u}_k \rangle = \langle u_k \rangle^2 + \langle u_k'^2 \rangle,$$

where $u'^2 = (u'_x)^2 + (u'_y)^2 + (u'_z)^2$. Therefore, to close the system [5], obtained by spatial averaging, it is necessary to obtain constitutive relations for the following quantities:

(1) Δp_{ki}.
(2) $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_i$.
(3) $\langle u_k'^2 \rangle$.

2.2. *Constitutive relations*

We begin by considering the gas bubbles moving as solid particles and we will assume that the bubbles have the same velocity inside the averaging volume. We will see that this assumption is implicit in the work of Geurst (1985a, b, 1986). This implies that

$$\langle u_G'^2 \rangle = 0. \quad [7]$$

We will use the average interfacial pressure to define a bulk pressure in the gas phase as follows:

$$\langle p_G \rangle = \frac{V}{a_i} \langle p_{Gi} \rangle_i = \frac{1}{a_i} \int_{a_i} p_{Gi} \, dS. \quad [8]$$

This definition will turn out to give the same results as the one used by Biesheuvel & van Wijngaarden (1984) and Wallis (1989). Interestingly, this notion of a bulk pressure has been used in the homogenization of Stokes flow around a bed of spheres by Lipton & Avellaneda (1990). This form of the bulk pressure gives

$$\Delta p_{Gi} = 0. \quad [9]$$

The determination of $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_i$ is more difficult. We shall ignore surface tension and therefore find $p_{Gi} = p_{Li}$, which implies that

$$\langle \mathbf{n}_G \cdot \mathbf{n}_z \Delta p'_{Gi} \rangle_i = -\langle \mathbf{n}_L \cdot \mathbf{n}_z \Delta p'_{Li} \rangle_i \quad [10]$$

since $\mathbf{n}_L = -\mathbf{n}_G$. Also, $p_{Gi} = p_{Li}$ combined with $\Delta p_{Gi} = 0$ gives

$$\langle p_G \rangle = \langle p_L \rangle + \Delta p_{Li}. \quad [11]$$

Below we present two derivations of $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_i$. The first derivation follows directly from the averaging procedure. The second derivation infers $\langle \mathbf{n}_G \cdot \mathbf{n}_z \Delta p'_{Gi} \rangle_i$ by a new interpretation of Voinov's (1973) computation of the force on a bubble.

The first derivation begins with the definition of $\langle \mathbf{n}_L \cdot \mathbf{n}_z \Delta p'_{Li} \rangle_i$. We have

$$\langle \mathbf{n}_L \cdot \mathbf{n}_z \Delta p'_{Li} \rangle_i = \frac{1}{V} \int_{a_i} \left[p_{Li} - \frac{V}{a_i} \langle p_{Li} \rangle_i \right] \mathbf{n}_L \cdot \mathbf{n}_z \, dS, \quad [12]$$

where the integration is taken on all of the bubble surfaces that are contained inside the averaging volume. Since $(V/a_i)\langle p_{Li} \rangle_i$ is a constant inside the averaging volume the second term only picks up contributions from the bubbles that cross the averaging volume. Indeed, one can show that

$$\frac{1}{V} \int_{a_i} \mathbf{n}_L \cdot \mathbf{n}_z \, dS = \frac{\partial \epsilon_G}{\partial z}.$$

This follows from Delhaye & Archard (1976, equation [4.5]). Therefore, we have

$$\langle \mathbf{n}_L \cdot \mathbf{n}_z \Delta p'_{Li} \rangle_i = \frac{1}{V} \int_{\tilde{a}_i} p_{Li} \mathbf{n}_L \cdot \mathbf{n}_z \, dS + \frac{1}{V} \int_{\bar{a}_i} p_{Li} \mathbf{n}_L \cdot \mathbf{n}_z \, dS - \frac{V}{a_i} \langle p_{Li} \rangle_i \frac{\partial \epsilon_G}{\partial z}, \quad [13]$$

where $\tilde{a}_i$ are all the bubble surfaces that do not cross the averaging volume and $\bar{a}_i$ are bubble surfaces that do cross the averaging volume. In appendix A it is shown that:

$$\frac{1}{V} \int_{\bar{a}_i} p_{Li} \mathbf{n}_L \cdot \mathbf{n}_z \, dS = \frac{\partial}{\partial z} \left(\epsilon_G \frac{V}{a_i} \langle p_{Li} \rangle_i \right) + \text{h.o.t.} \quad [14]$$

and

$$\frac{1}{V} \int_{\bar{a}_i} p_{Li} \mathbf{n}_L \cdot \mathbf{n}_z \, dS = \epsilon_G \frac{\partial \langle p_L \rangle}{\partial z} - \frac{\epsilon_G}{\frac{4}{3}\pi R^3} \int_B p'_{Li} \mathbf{n}_G \cdot \mathbf{n}_z \, dS + \text{h.o.t.} \quad [15]$$

By h.o.t. we mean terms that tend to zero as $R \to 0$ for fixed ϵ_G and $O(\epsilon_G^2)$terms. Here R is the bubble radius. By letting $R \to 0$ we mean that we are considering only length scales in the averaged equations that are much bigger than the bubble size. In [15] p'_{Li} is the disturbance in the pressure field caused by the introduction of one sphere into the bubbly mixture. Substituting [14] and [15] into [13] gives

$$\langle \mathbf{n}_L \cdot \mathbf{n}_z \Delta p'_{Li} \rangle_i = -\frac{\epsilon_G}{\frac{4}{3}\pi R^3} \int_B p'_{Li} \mathbf{n}_G \cdot \mathbf{n}_z \, dS + \epsilon_G \frac{\partial}{\partial z} \Delta p_{Li} + \text{h.o.t.}, \quad [16]$$

where we have used [3] to eliminate $(V/a_i)\langle p_{ki} \rangle_i$. In appendix B it is shown that

$$\int_B p'_{Li} \mathbf{n}_G \cdot \mathbf{n}_z \, dS = \tfrac{2}{3} \pi R^3 \rho_L \left(\frac{D_G u_G}{Dt} - \frac{D_L \langle u_L \rangle}{Dt} \right) + O(R^5),$$

where R is the radius of the sphere and

$$\frac{D_k}{Dt} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial z}.$$

Using the last result combined with [7] and [16] gives us

$$\langle \mathbf{n}_L \cdot \mathbf{n}_z \Delta p'_{Li} \rangle_i = \tfrac{1}{2} \rho_L \epsilon_G \left(\frac{D_L \langle u_L \rangle}{Dt} - \frac{D_G \langle u_G \rangle}{Dt} \right) + \epsilon_G \frac{\partial \Delta p_{Li}}{\partial z} + \text{h.o.t.} \quad [17]$$

There is an alternative way to infer [17]; however, it does not fit in directly with the averaging method. We include it here because it provides a simpler and perhaps more physical way to obtain

[17]. This is a new interpretation of Voinov's (1973) result. Using his result Pauchon & Banerjee (1986) showed that the equation of motion of a single bubble in one space dimension in an arbitrary potential flow is given by

$$\rho_G \frac{D_G u_G}{Dt} = -\frac{\partial p_L}{\partial z} - \tfrac{1}{2}\rho_L\left(\frac{D_G u_G}{Dt} - \frac{D_L u_L}{Dt}\right) + O(R^5). \qquad [18]$$

In [18] the first term on the r.h.s. corresponds to the virtual buoyancy and the second term is commonly called the virtual mass acceleration. Clearly, from [5] this cannot be $\langle \mathbf{n}_G \cdot \mathbf{n}_z \Delta p_{Gi} \rangle_i$. We identify [18] as the averaged gas momentum equation, as given by [5], using the fact that $\Delta p_{Gi} = 0$ and $p_L = p_G - \Delta p_{Li}$, we get

$$\rho_G \epsilon_G \frac{D_G u_G}{Dt} + \epsilon_G \frac{\partial p_G}{\partial z} = -\tfrac{1}{2}\rho_L \epsilon_G \left(\frac{D_G u_G}{Dt} - \frac{D_L u_L}{Dt}\right) + \epsilon_G \frac{\partial \Delta p_{Li}}{\partial z} + \text{h.o.t.}$$

Thus, it is apparent that the symmetrical virtual mass acceleration should be

$$\langle \mathbf{n}_G \cdot \mathbf{n}_z \Delta p_{Gi} \rangle_i = \tfrac{1}{2}\rho_L \epsilon_G \left(\frac{D_G \langle u_G \rangle}{Dt} - \frac{D_L \langle u_L \rangle}{Dt}\right) - \epsilon_G \frac{\partial \Delta p_{Li}}{\partial z} + \text{h.o.t.},$$

which is equivalent to [17], in view of [10]. The second term in [17] seems to have been overlooked by previous investigators.

We turn our attention to the velocity distribution term in the liquid phase. In appendix B we consider the motion of a single sphere in a potential flow with a velocity gradient. Straightforward calculations show that

$$\int_{V_L} u_L'^2 \, dV = \frac{2\pi R^3}{3} u_r^2 + O(R^5),$$

where $u_r = \langle u_G \rangle - \langle u_L \rangle$. Using [7] we infer

$$\langle u_L'^2 \rangle = \tfrac{1}{2}\epsilon_G u_r^2 + \text{h.o.t.} \qquad [19]$$

Finally, we turn our attention to the pressure difference term. We have from appendix B that

$$\int_B (p_{Li} - \langle p_L \rangle)\, dS = -\pi R^2 \rho_L u_r^2 + O(R^5).$$

From which we infer, again using [7], that

$$\Delta p_{Li} = -\tfrac{1}{4}\rho_L u_r^2 + \text{h.o.t.} \qquad [20]$$

Substitution of [20] into [11] gives

$$\langle p_G \rangle = \langle p_L \rangle - \tfrac{1}{4}\rho_L u_r^2 + \text{h.o.t.} \qquad [21]$$

Equation [21] is well known (e.g. Biesheuvel & van Wijngaarden 1984; Wallis 1989). Substitution of [20] into [17] offers some simplification:

$$\langle \mathbf{n}_G \cdot \mathbf{n}_z \Delta' p_{Gi} \rangle_i = -\langle \mathbf{n}_L \cdot \mathbf{n}_z \Delta' p_{Li} \rangle = \epsilon_G \frac{\rho_L}{2}\left[\frac{\partial u_r}{\partial t} + \frac{\partial u_r \langle u_G \rangle}{\partial z}\right] + \text{h.o.t.}, \qquad [22]$$

where $u_r = \langle u_G \rangle - \langle u_L \rangle$. Equations [7], [9] and [19]–[22] complete the model. Their substitution into [5] and ignoring the h.o.t. terms yields the averaged momentum equations valid in the dilute limit. We find

$$\rho_L\left[\frac{\partial \epsilon_L \langle u_L \rangle}{\partial t} + \frac{\partial}{\partial z}\left(\epsilon_L \langle u_L \rangle^2 + \tfrac{1}{2}\epsilon_L \epsilon_G u_r^2\right)\right] + \epsilon_L \frac{\partial \langle p_L \rangle}{\partial z} + \tfrac{1}{4}\rho_L u_r^2 \frac{\partial \epsilon_L}{\partial z} = \frac{\epsilon_G \rho_L}{2}\left(\frac{\partial u_r}{\partial t} + \frac{\partial u_r \langle u_G \rangle}{\partial z}\right) \qquad [23a]$$

and

$$\rho_G\left(\frac{\partial \epsilon_G \langle u_G \rangle}{\partial t} + \frac{\partial}{\partial z}\epsilon_G \langle u_G \rangle^2\right) + \epsilon_G \frac{\partial}{\partial z}\left(\langle p_L \rangle - \tfrac{1}{4}\rho_L u_r^2\right) = -\frac{\epsilon_G \rho_L}{2}\left(\frac{\partial u_r}{\partial t} + \frac{\partial u_r \langle u_G \rangle}{\partial z}\right). \qquad [23b]$$

In the next section, we turn our attention to the variational method. Starting from the constitutive relation for the velocity perturbation term we shall show, by use of this single

constitutive relation, that we can obtain all the constitutive relations characterizing the pressure field. This will, in the dilute limit, reconcile the averaging and variational approach. In the nondilute case the variational approach will lead to the form of the virtual mass acceleration with a single unknown coefficient function of the void fraction.

3. THE VARIATIONAL APPROACH

Traditionally, variational principles have been associated with problems of Hamiltonian dynamics. In continuum mechanics, the extension of Hamilton's principle is straightforward in a Lagrangian description. The difficulty arises when the Eulerian description is used. The main ideas which resolved this issue are due to Clebsch (1859) and Lin (1963). A discussion of the problem is given by Seliger & Whitman (1968), and the methodology is explained in Whitman (1974).

3.1. Formulation of the Lagrangian

We follow the approach of Geurst (1985b), however since we consider the incompressible case, there will be slight differences, namely the introduction of an additional constraint. We shall also show that there is an implicit assumption in Geurst's derivation, which is that all of the gas bubbles in the averaging volume have the same velocity. The Lagrangian is defined as the difference between the kinetic and potential energy of the system. Since the two phases are considered incompressible, the Lagrangian reduces to the sum of the averaged kinetic energies of the two phases per unit of total volume:

$$\bar{K} = \sum_k \frac{1}{V_k} \int_{V_k} \tfrac{1}{2} \rho_k \mathbf{u}_k \cdot \mathbf{u}_k \, \mathrm{d}V$$

and

$$\bar{K} = \sum_k \tfrac{1}{2} \epsilon_k \rho_k \langle \mathbf{u}_k \cdot \mathbf{u}_k \rangle,$$

where $\bar{K}$ is the kinetic energy density of the averaging volume. As before, we assume that the flow is one-dimensional in the z-direction, so that

$$\langle \mathbf{u}_\mathrm{L} \rangle = \langle u_\mathrm{L} \rangle \mathbf{n}_z$$

and

$$\bar{K} = \frac{1}{2} \sum_k \epsilon_k \rho_k (\langle u_k \rangle^2 + \langle u_k'^2 \rangle). \tag{24}$$

It is clear from [24] that $\langle u_\mathrm{G}'^2 \rangle$ must be considered, although one is tempted to argue that since ρ_G is small its contribution will be small. However, we shall see below that $\langle u_\mathrm{G}'^2 \rangle$ can play an important role in the $\langle u_\mathrm{L}'^2 \rangle$ term. To see this, we first write the liquid velocity as

$$\mathbf{u}_\mathrm{L} = \nabla(\langle u_\mathrm{L} \rangle z + \Psi).$$

This must be true since we have taken the fluid velocity to be governed by potential flow. Assuming the bubbles move like rigid spheres the boundary condition on the jth bubble is

$$\nabla \Psi \cdot \mathbf{n}_\mathrm{G} = (\mathbf{u}_{\mathrm{G}j} - \langle u_\mathrm{L} \rangle \mathbf{n}_z) \cdot \mathbf{n}_\mathrm{G}, \tag{25}$$

where $\mathbf{n}_\mathrm{G}$ and $\mathbf{u}_{\mathrm{G}j}$ are the outward drawn normal and velocity of the jth bubble, respectively. The Reynolds stress term is then

$$\langle u_\mathrm{L}'^2 \rangle = \frac{1}{V_\mathrm{L}} \int_{V_\mathrm{L}} |\mathbf{u}_\mathrm{L}'|^2 \, \mathrm{d}V = \frac{1}{V_\mathrm{L}} \int_{V_\mathrm{L}} |\nabla \Psi|^2 \, \mathrm{d}V. \tag{26}$$

Since the flow is incompressible then $\nabla^2 \Psi = 0$. The equation for Ψ is linear and the solution must be of the form

$$\Psi = \sum_{j=1}^{N} (\mathbf{u}_{\mathrm{G}j} - \langle u_\mathrm{L} \rangle \mathbf{n}_z) \cdot \boldsymbol{\Psi}_j, \tag{27}$$

where N is the number of bubbles in the averaging volume and Ψ_j is a vector-valued function that is entirely determined by the position of the bubbles. Substituting [27] into [26] produces:

$$\langle u_L'^2 \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} (\mathbf{u}_{Gj} - \langle u_L \rangle \mathbf{n}_z)^T A_{ij} (\mathbf{u}_{Gi} - \langle u_L \rangle \mathbf{n}_z), \quad [28]$$

where the superscript T denotes transpose and A_{ij} is a 3×3 matrix for each i and j. The matrices A_{ij} depend only on the position of the bubbles. We would like to express [28] in terms of the averaged quantities, $\langle \mathbf{u}_G \rangle$ and $\langle u_L \rangle$, since that is not possible because of all the cross terms we assume, instead, that the bubbles have the same velocity in the averaging volume and move in the same direction as the fluid, therefore we take

$$\mathbf{u}_{Gj} = \langle u_G \rangle \mathbf{n}_z \quad \forall j,$$

which of course implies that

$$\langle u_G'^2 \rangle = 0. \quad [29]$$

This assumption was made implicitly by Geurst (1985a, b, 1986). Substitution of this into [28] gives

$$\langle u_L'^2 \rangle = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{n}_z^T A_{ij} \mathbf{n}_z \right) (\langle u_G \rangle - \langle u_L \rangle)^2. \quad [30]$$

The function multiplying $(\langle u_G \rangle - \langle u_L \rangle)^2$ is solely determined by the configuration of the bubbles. Geurst's assumption was that [30] can be written as

$$\langle u_L'^2 \rangle = m(\epsilon_G)(\langle u_G \rangle - \langle u_L \rangle)^2 = m(\epsilon_G) u_r^2. \quad [31]$$

The coefficient $m(\epsilon_G)$ is a phenomenological relation that depends on the details of the flow configuration. It should not be confused with the virtual mass coefficient. It is another phenomenological function that is quite different for nondilute bubbly flows. Wallis (1989) calls this the exertia, Smereka & Milton (1991) call it Reynolds stress coefficient. We know from [19] that in the dilute limit

$$m(\epsilon_G) = \tfrac{1}{2} \epsilon_G.$$

For the sake of convenience, we will now drop all symbols denoting averaging. From [24], [29] and [31], the kinetic energy density of the two-phase mixture is

$$K = \tfrac{1}{2} \epsilon_L \rho_L u_L^2 + \tfrac{1}{2} \epsilon_L \rho_L m(\epsilon_G) u_r^2 + \tfrac{1}{2} \epsilon_G \rho_G u_g^2. \quad [32]$$

In the case of compressible flow, the Lagrangian is equal to the kinetic energy since there is no potential energy. However, no information concerning mass conservation is embodied in this expression. Thus, we must add to our Lagrangian, constraints expressing conservation of the mass. Following Geurst (1985b) we define the following constraints:

$$L_1 = \Phi_L \rho_L \left(\frac{\partial \epsilon_L}{\partial t} + \frac{\partial \epsilon_L u_L}{\partial z} \right) + \Phi_G \rho_G \left(\frac{\partial \epsilon_G}{\partial t} + \frac{\partial \epsilon_G u_G}{\partial z} \right), \quad [33]$$

where Φ_L and Φ_G are called Lagrange multipliers (functions of z and t only). In addition, we shall treat the individual volume fractions as independent variables, thus it is necessary to add the following constraint, where Φ_α is also a Lagrange multiplier:

$$L_2 = \Phi_\alpha (\epsilon_G + \epsilon_L). \quad [34]$$

A similar constraint was used by Bedford & Drumheller (1978). Thus, our Lagrangian density is expressed as

$$L = K + L_1 + L_2 \quad [35]$$

and the variational principle reads:

$$\delta \int\int L \, dz \, dt = 0, \quad [36]$$

where δ is the differentiation symbol in the space of dependent variables (ϵ_G, ϵ_L, u_G and u_L). Application of the variational principle [36] for the four variables u_L, ϵ_L, u_G and ϵ_G yields:

$$\delta u_L: \quad \epsilon_L \rho_L u_L - \epsilon_L m(\epsilon_G)\rho_L u_r - \epsilon_L \rho_L \frac{\partial \Phi_L}{\partial z} = 0, \tag{37a}$$

$$\delta \epsilon_L: \quad \tfrac{1}{2}\rho_L u_L^2 + \tfrac{1}{2} m(\epsilon_G)\rho_L u_r^2 + \Phi_\alpha - \rho_L\left(\frac{\partial \Phi_L}{\partial t} + u_L \frac{\partial \Phi_L}{\partial z}\right) = 0, \tag{37b}$$

$$\delta u_G: \quad \epsilon_G \rho_G u_G + \epsilon_L m(\epsilon_G)\rho_L u_r - \epsilon_G \rho_G \frac{\partial \Phi_G}{\partial z} = 0 \tag{38a}$$

and

$$\delta \epsilon_G: \quad \tfrac{1}{2}\rho_G u_G^2 + \tfrac{1}{2}\epsilon_L m'(\epsilon_G)\rho_L u_r^2 + \Phi_\alpha - \rho_G\left(\frac{\partial \Phi_G}{\partial t} + u_G \frac{\partial \Phi_G}{\partial z}\right) = 0, \tag{38b}$$

where the $'$ stands for differentiation with respect to ϵ_G. Combining [37a, b] and [38a, b] to eliminate $\partial \Phi_k / \partial z$ yields:

$$\tfrac{1}{2}\epsilon_L \rho_L u_L^2 - \epsilon_L \rho_L m(\epsilon_G) u_r u_L - \tfrac{1}{2}\epsilon_L m(\epsilon_G)\rho_L u_r^2 - \epsilon_L \Phi_\alpha + \epsilon_L \rho_L \frac{\partial \Phi_L}{\partial t} = 0 \tag{39a}$$

and

$$\tfrac{1}{2}\epsilon_G \rho_G u_G^2 + \epsilon_L m(\epsilon_G)\rho_L u_r u_G - \tfrac{1}{2}\epsilon_L \epsilon_G m'(\epsilon_G)\rho_L u_r^2 - \epsilon_G \Phi_\alpha + \epsilon_G \rho_G \frac{\partial \Phi_G}{\partial t} = 0. \tag{39b}$$

3.2. Equations of motion

To derive the equations of motion, we eliminate Φ_G and Φ_L using [39a, b] and [37a] and [38a] to obtain:

$$\rho_L\left(\frac{\partial}{\partial t}\epsilon_L u_L + \frac{\partial}{\partial z}\epsilon_L u_L^2 + \frac{\partial}{\partial z}\epsilon_L m(\epsilon_G) u_r^2\right) - \epsilon_L \frac{\partial \Phi_\alpha}{\partial z} + \tfrac{1}{2}\epsilon_L \rho_L m'(\epsilon_G) u_r^2 \frac{\partial \epsilon_L}{\partial z}$$
$$= +\rho_L\left(\frac{\partial}{\partial t}\epsilon_L m(\epsilon_G) u_r + \frac{\partial}{\partial z}\epsilon_L m(\epsilon_G) u_r u_G + \epsilon_L m(\epsilon_G) u_r \frac{\partial u_G}{\partial z}\right). \tag{40a}$$

Similarly,

$$\rho_G\left(\frac{\partial}{\partial t}\epsilon_G u_G + \frac{\partial}{\partial z}\epsilon_G u_G^2\right) - \epsilon_G \frac{\partial}{\partial z}\left(\Phi_\alpha + \tfrac{1}{2}\epsilon_L \rho_L m'(\epsilon_G) u_r^2\right)$$
$$= -\rho_L\left(\frac{\partial}{\partial t}\epsilon_L m(\epsilon_G) u_r + \frac{\partial}{\partial z}\epsilon_L m(\epsilon_G) u_r u_G + \epsilon_L m(\epsilon_G) u_r \frac{\partial u_G}{\partial z}\right). \tag{40b}$$

Comparing [40a] with the averaged momentum equation [5] for phase L, we see that

$$\Phi_\alpha = -p_L,$$

and further comparison reveals that:

(1) The velocity distribution term is, as postulated,

$$\langle u_L'^2 \rangle = m(\epsilon_G) u_r^2. \tag{41a}$$

(2) The phasic pressure difference terms are

$$\Delta p_{L_i} = -\tfrac{1}{2}\epsilon_L m'(\epsilon_G)\rho_L u_r^2 \tag{41b}$$

and

$$\Delta p_{G_i} = 0. \tag{41c}$$

(3) The symmetric virtual mass acceleration is

$$\langle \mathbf{n}_G \cdot \mathbf{n}_z \Delta p'_{G_i} \rangle_i = p_L\left(\frac{\partial}{\partial t}\epsilon_L m(\epsilon_G) u_r + \frac{\partial}{\partial z}\epsilon_L m(\epsilon_G) u_r u_G + \epsilon_L m(\epsilon_G) u_r \frac{\partial u_G}{\partial z}\right). \tag{41d}$$

Equation [41d] confirms the result obtained previously by Geurst (1986) who also shows that this acceleration is objective. A result similar to [41b] was also obtained by Lhuillier (1985). It is interesting to notice that since $-\Phi_\alpha$ is the average pressure then L_2 is a work term since ϵ_L and ϵ_G are akin to volumes. If we compare these equations with those obtained by Wallis (1990), we see that if his coefficient E plays the same role as our coefficient m, [41b] corresponds to equation [48] in Wallis's work, and if p_s (the "interfacial stress tensor") is identified to the dispersed phase pressure, equation [47] in Wallis (1991) is identical to [41a]. This shows that the two approaches lead to the same conclusion. Wallis (1990) does not propose a form for the mutual interfacial force defined as $\mathbf{f}_{12}$. It is possible that if it includes a term equivalent to a dispersed phase pressure gradient, his equations of motion [37] and [38] will not be essentially different from the one proposed here.

3.3. Generalized Bernoulli equations

From [39a, b] we seek to derive Bernoulli type equations for each phase. These equations should, however, be independent of the phases, except for the velocity perturbation term which arises only in the continuous phase. In addition, these should reveal the energy exchange term corresponding to the "symmetric" virtual mass acceleration. Substituting $-\Phi_\alpha$ with p_L and using [41a, b] leads to:

$$\tfrac{1}{2}\rho_L\epsilon_L u_1^2 + \tfrac{1}{2}\rho_L\epsilon_L\langle u_L'^2\rangle - \epsilon_L m(\epsilon_G)\rho_L u_r u_G + \epsilon_L p_L + \rho_L\epsilon_L\frac{\partial\Phi_L}{\partial t} = 0 \qquad [42a]$$

and

$$\tfrac{1}{2}\rho_G\epsilon_G u_G^2 + \epsilon_L m(\epsilon_G)\rho_L u_r u_G + \epsilon_G p_G + \epsilon_G\rho_G\frac{\partial\Phi_G}{\partial t} = 0. \qquad [42b]$$

These are the Bernoulli equations for both phases. As anticipated, the velocity distribution term occurs only in the continuous phase. Interestingly, [42a, b] suggest that the energy exchange term is precisely

$$\epsilon_L m(\epsilon_G)\rho_L u_r u_G. \qquad [43]$$

This energy transfer from the gas to the liquid is manifested by a lower pressure in the liquid phase due to the Bernoulli effect, see [41b], and by the additional kinetic energy associated with the liquid velocity fluctuations in the liquid. Thus, [42a, b] clarify the connection between the liquid velocity perturbation, the symmetric interfacial momentum transfer term and the virtual mass force. Wallis (1990) also found similar Bernoulli equations and examined their consequences.

3.4. Equivalence of the averaging and variational approaches

To begin, we note that when $\epsilon_G \to 0$, $\epsilon_L \to 1$, $m(\epsilon_G) \to \epsilon_G/2$ and $m'(\epsilon_G) \to 1/2$. Therefore, it follows that as $\epsilon_G \to 0$, [41a] and [41b] become

$$\langle u_L'^2\rangle = \tfrac{1}{2}\epsilon_G u_r^2 \qquad [44a]$$

and

$$\Delta p_{Li} = -\tfrac{1}{4}\rho_L u_r^2, \qquad [44b]$$

respectively, which are in agreement with [19] and [20]. Next we write [41d] as

$$\langle \mathbf{n}_G\cdot\mathbf{n}_z\,\Delta p_{Gi}'\rangle_i = \rho_L m(\epsilon_G)\epsilon_L\left(\frac{\partial u_r}{\partial t} + \frac{\partial u_r u_G}{\partial z}\right) + \rho_L[m(\epsilon_G) - \epsilon_L\epsilon_G m'(\epsilon_G)]u_r\frac{\partial u_G}{\partial z}.$$

Notice that in the dilute limit $m(\epsilon_G) = \frac{1}{2}\epsilon_G$, so the above expression becomes

$$\langle \mathbf{n}_G\cdot\mathbf{n}_z\,\Delta p_{Gi}'\rangle_i = \tfrac{1}{2}\rho_L\epsilon_G\left(\frac{\partial u_r}{\partial t} + \frac{\partial u_r u_G}{\partial z}\right), \qquad [44c]$$

in agreement with [22]. Therefore, [44a–c] show that the averaging approach yields identical equations to the variational approach in the dilute limit. This means that the dilute limit of [40a, b] is identical to [23].

3.5. *Discussion*

It has been established by Geurst (1986) that the form of the "symmetric" virtual mass acceleration given by [41d] is objective when written in three-dimensional form. Geurst's result is valid for any $m(\epsilon_G)$, therefore if we take $m(\epsilon_G) = \frac{1}{2}\epsilon_G$ we infer that the three-dimensional form of [44c] is also objective. There is a direct way to see that the "symmetric" virtual mass must be objective from the work of Gurtin & Struthers (1991). They show that any interfacial force that acts on an interface of zero mass must be objective. In this case, the interfacial force comes from the interfacial pressure and we assume the interface to have no mass.

The discrepancy between [44c] and the virtual mass acceleration proposed by Pauchon & Banerjee (1986) or Drew & Lahey (1987) amongst others, shows that the symmetric virtual mass acceleration is actually the sum of the acceleration commonly used,

$$a_{VM} = \tfrac{1}{2}\rho_L \epsilon_G \left(\frac{\partial u_r}{\partial t} + u_G \frac{\partial u_G}{\partial z} - u_L \frac{\partial u_L}{\partial z} \right), \qquad [45]$$

plus the contribution of Δp_{Li} as explicited in [17]. The difference between [44c] and [45] arises from the liquid pressure gradient (or virtual buoyancy term) which enters partially in [44c] and [22].

The results contained in [41a–d] show that the form of the constitutive relationships is determined solely from the expression for the Reynolds stress term. Thereby indicating that the constitutive laws for the phasic pressure difference Δp_{Li}, the Reynolds stress $\langle u_L'^2 \rangle$ and the "symmetric" virtual mass term $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p_{ki}' \rangle_i$ are intimately related. Erroneous conclusions may be reached if these terms are approached independently.

To include bubbly flows with higher void fractions, we may use the constitutive relations [41a–d]. However, the problem of determining the form of $m(\epsilon_G)$ remains. Wallis (1989) and Smereka & Milton (1991) have related $m(\epsilon_G)$ to the effective conductivity of a composite material containing a uniform conductor and insulating spherical particles. This allows $m(\epsilon_G)$ to be determined for nondilute mixtures. The importance of this function will be discussed in the next section.

4. PROPERTIES OF THE EQUATIONS OF MOTION

In this section we shall outline several important properties of equations describing the flow of two-phase dispersed mixtures. To begin this analysis we reduce the model to a set of two equations, using the drift flux concept, see Zuber (1964), Wallis (1969) or Pauchon (1989).

4.1. *The reduced system*

Adding both mass conservations [1a] gives:

$$\frac{\partial j_0}{\partial z} = 0, \quad j_0 = \epsilon_G u_G + \epsilon_L u_L, \qquad [46]$$

since $\epsilon_G + \epsilon_L = 1$. This means that j_0 is only a function of t, and determined by the boundary conditions. We shall take j_0 to be a constant and consider the transformation

$$z^* = z - j_0 t$$

and

$$t^* = t.$$

Then the velocities transform as:

$$u_k^* = u_k - j_0,$$

more specifically

$$u_L^* = -\epsilon_G u_r$$

and

$$u_G^* = \epsilon_L u_r.$$

The mass conservation equations transform as

$$\frac{\partial \epsilon_k}{\partial t^*} + \frac{\partial \epsilon_k u_k^*}{\partial z^*} = 0.$$

Substitution of u_L^* and u_G^* into the transformed equation shows that both conservations become identical:

$$\frac{\partial \epsilon_G}{\partial t^*} + \frac{\partial J}{\partial z^*} = 0, \quad J = \epsilon_G \epsilon_L u_r, \tag{47}$$

where J is known as the drift flux. In the following analysis, we propose to drop the * sign and we set:

$$\epsilon_G = \epsilon, \quad \epsilon_L = 1 - \epsilon.$$

Substituting the transformed velocities into the expression for the kinetic energy gives:

$$K = \tfrac{1}{2}\Gamma(\epsilon)J^2, \tag{48}$$

where

$$\Gamma(\epsilon) = \frac{\rho_L}{1-\epsilon} + \frac{\rho_G}{\epsilon} + \frac{\rho_L m(\epsilon)}{(1-\epsilon)\epsilon^2}.$$

The new Lagrangian can be written as

$$L = K - \Phi\left(\frac{\partial \epsilon}{\partial t} + \frac{\partial J}{\partial z}\right), \tag{49}$$

using $-\Phi$ instead of Φ as a matter of convenience only. The variational principle reads

$$\delta \int L \, dz \, dt = 0.$$

Computing the variation of L we find:

$$\delta\epsilon: \quad \tfrac{1}{2}\Gamma_\epsilon(\epsilon)J^2 + \Phi_t = 0 \tag{50a}$$

and

$$\delta J: \quad J\Gamma + \Phi_z = 0, \tag{50b}$$

where the index ϵ indicates differentiation with respect to the variable ϵ.

One can eliminate Φ to obtain an equation for J, which combined with the constraint of mass conservation yields the reduced set of equations:

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial J}{\partial z} = 0 \tag{51a}$$

and

$$\frac{\partial}{\partial t} J\Gamma - \frac{\partial}{\partial z}\left(\tfrac{1}{2}\Gamma_\epsilon J^2\right) = 0. \tag{51b}$$

The first equation expresses mass conservation and the second equation is a slip equation, written in a frame of reference moving at the velocity j_0. In [51a, b] we recognize the form of the slip equation derived by Pauchon (1989) in the case of a single pressure model (by setting $m = 0$). Note that in [51a, b] the presence of the virtual mass does not modify the structure of the slip equation.

4.2. Hamiltonian formalism

We now propose to derive the equations of motion in yet another way in order to clarify the Hamiltonian formalism of the equations of motion. The first part of this derivation is inspired by the work of Seliger & Whitham (1968).

From the variational method, we see that $J\Gamma = -\Phi_z$. Substituting this into the expression for the kinetic energy and using integration by parts on the constraint yield an equivalent Lagrangian:

$$L^* = -\frac{1}{2}\frac{\Phi_z^2}{\Gamma} + \epsilon\Phi_t. \tag{52}$$

The important point here is that by the use of a potential representation for ΓJ, we can eliminate the explicit appearance of the constraints. One can readily verify that

$$\delta \int L^* \, dt \, dz = 0$$

yields the equations of motion. We shall use L^* to derive the Hamiltonian.

To form the Hamiltonian, one can simply follow the natural generalization of Hamiltonian mechanics from discrete systems to continuous systems (e.g. Leech 1965). The generalized velocity is Φ_t, and consequently the generalized momentum is

$$\frac{\partial L^*}{\partial \Phi_t} = \epsilon$$

and the Hamiltonian density is given by

$$H = \frac{\partial L^*}{\partial \Phi_t}\Phi_t - L^*.$$

Thus, the Hamiltonian is

$$\mathscr{H} = \int_{-\infty}^{+\infty} H \, dz = \int_{-\infty}^{+\infty} \frac{\Phi_z^2}{2\Gamma} \, dz. \tag{53}$$

The canonical variables are (Φ, ϵ), consequently the equations of motion are

$$\Phi_t = \frac{\delta \mathscr{H}}{\delta \epsilon} = \frac{-\Gamma_\epsilon}{2\Gamma^2}\Phi_z^2 \tag{54a}$$

and

$$\epsilon_t = -\frac{\delta \mathscr{H}}{\delta \Phi} = \left(\frac{\Phi_z}{\Gamma}\right)_z. \tag{54b}$$

Unfortunately, these equations are written in terms of a potential, which is not a physically interesting variable. To transform the problem to physically relevant quantities, it is useful to introduce the Poisson bracket:

$$\{\mathscr{F}, \mathscr{G}\} = \int_{-\infty}^{+\infty} \left(\frac{\delta \mathscr{F}}{\delta \Phi}\frac{\delta \mathscr{G}}{\delta \epsilon} - \frac{\delta \mathscr{G}}{\delta \Phi}\frac{\delta \mathscr{F}}{\delta \epsilon}\right) dz, \tag{55}$$

where $\mathscr{F}$ and $\mathscr{G}$ are functionals of Φ and ϵ [see, for example, Morisson (1981) or Holm & Kupershmidt (1983)]. The equations of motion are

$$\mathscr{F}_t = \{\mathscr{F}, \mathscr{H}\}.$$

We shall now use the Poisson bracket to transform the equations of motion. We have

$$\Gamma J = -\Phi_z = M, \tag{56}$$

then the Hamiltonian becomes

$$\mathscr{H} = \int_{-\infty}^{+\infty} \frac{M^2}{2\Gamma} \, dz \tag{57}$$

and by the chain rule for functional derivatives (Morisson 1981) we have

$$\frac{\delta \mathscr{F}}{\delta M} = \frac{\partial}{\partial z}\frac{\partial \mathscr{F}}{\delta \Phi}.$$

Substituting this into the Poisson brackets and using integration by parts we find

$$\{\mathscr{F}, \mathscr{G}\} = -\int_{-\infty}^{+\infty} \left(\frac{\delta \mathscr{F}}{\delta M} \frac{\partial}{\partial z} \frac{\delta \mathscr{G}}{\delta \epsilon} + \frac{\delta \mathscr{F}}{\delta \epsilon} \frac{\partial}{\partial z} \frac{\delta \mathscr{G}}{\delta M} \right) dz. \quad [58]$$

Thus, we can write the equations of motion as

$$M_t = \{M, \mathscr{H}\} = -\frac{\partial}{\partial z} \frac{\delta \mathscr{H}}{\delta \epsilon} = \frac{\partial}{\partial z} \left(\frac{\Gamma_\epsilon M^2}{2\Gamma^2} \right) \quad [59a]$$

and

$$\epsilon_t = \{\epsilon, \mathscr{H}\} = -\frac{\partial}{\partial z} \frac{\delta \mathscr{H}}{\delta M} = -\frac{\partial}{\partial z} \left(\frac{M}{\Gamma} \right). \quad [59b]$$

We note that the bracket for these equations has a slight similarity to the Poisson bracket for compressible flow (see Holm *et al.* 1985).

4.3. Hyperbolicity

We write our equations of motion as

$$\begin{pmatrix} \epsilon \\ M \end{pmatrix}_t + \begin{pmatrix} -\dfrac{M\Gamma_\epsilon}{\Gamma^2} & \dfrac{1}{\Gamma} \\ -\dfrac{M^2}{2} \left(\dfrac{\Gamma_\epsilon}{\Gamma^2} \right)_\epsilon & -\dfrac{M\Gamma_\epsilon}{\Gamma^2} \end{pmatrix} \begin{pmatrix} \epsilon \\ M \end{pmatrix}_z = 0. \quad [60]$$

The eigenvalues of the above matrix determine whether or not the model is hyperbolic. We find:

$$\lambda^\pm = \frac{M\Gamma_\epsilon}{\Gamma^2} \left[-1 \pm \left(\frac{2\Gamma_\epsilon^2 - \Gamma\Gamma_{\alpha\alpha}}{2\Gamma_\epsilon^2} \right)^{1/2} \right]. \quad [61]$$

Therefore, the model will be hyperbolic if

$$2\Gamma_\epsilon^2 - \Gamma\Gamma_{\alpha\alpha} > 0. \quad [62]$$

This result is surprisingly simple, considering that it applies to a broad class of two-phase flow models. To investigate the consequences of the above condition, we take

$$m(\epsilon) = c_0 \epsilon (1 + c_1 \epsilon), \quad [63]$$

then one finds:

$$2\Gamma_\epsilon^2 - \Gamma\Gamma_{\alpha\alpha} = -\frac{2\rho_L (1 + c_0 + c_0 c_1)(\rho_G + c_0 \rho_L)}{\epsilon^3 (1 - \epsilon)^3}. \quad [64]$$

Thus, we see that if the virtual mass is neglected (set $c_0 = 0$), then the model is not hyperbolic. This is consistent with the fact that the single-pressure model of two-phase flow is ill-posed. For spherical particles, we know that $c_0 = 1/2$. Therefore, the hyperbolicity condition is

$$c_1 + 3 < 0. \quad [65]$$

As mentioned previously, the function $m(\epsilon_G)$ has been studied. This work shows that for random isotropic distributions of bubbles $c_1 = 0$, which indicates that the model is not hyperbolic. However, Smereka & Milton (1991) show that if the bubbles cluster, the resulting equations are found to be hyperbolic.

4.4. Lyapunov stability

In this section it will be proved that the condition for Lyapunov stability coincides with the hyperbolicity condition. The basic ideas as well as an overview of the subject and its applications can be found in Holm *et al.* (1985). The first step is to find as many conserved functionals as possible. For example, mass, momentum and energy. We then take an arbitrary linear combination of these, which we call $\mathscr{F}$. Then one must show that there are solutions of the equations of motion which are critical points of $\mathscr{F}$. The more general one can make $\mathscr{F}$, the wider the class of solutions one can study. After ensuring $\delta\mathscr{F} = 0$ for some solutions, one computes the second variation $\delta^2\mathscr{F}$

for these critical solutions. If one can show that $\delta^2\mathscr{F}$ is definite in sign, then $\delta^2\mathscr{F}$ can be used as a norm to measure the distance in the space of variables, between a given solution and the critical solution. Further, since $\mathscr{F}$ is a conserved quantity, this norm can be controlled by the initial condition. Thus, linear Lyapunov stability is established for critical solutions. To begin, we list the constants of motion:

$$\mathscr{M} = \int_{-\infty}^{+\infty} M \, dz, \tag{66a}$$

$$\mathscr{E} = \int_{-\infty}^{+\infty} \epsilon \, dz, \tag{66b}$$

$$\mathscr{H} = \int_{-\infty}^{+\infty} \frac{M^2}{2\Gamma} \, dz \tag{66c}$$

and

$$\mathscr{R} = \int_{-\infty}^{+\infty} M\epsilon \, dz. \tag{66d}$$

The first two follow from the conservative form of the equations of motion, the next is a statement of energy conservation and the last is obtained using the translation invariance of the equations of motion and Noether's theorem (see Geurst 1985b). One may verify that they are constants of the motion by using the Poisson bracket, e.g.

$$\mathscr{H}_t = \{\mathscr{H}, \mathscr{H}\} = 0 \tag{67a}$$

and

$$\mathscr{R}_t = \{\mathscr{R}, \mathscr{H}\} = 0. \tag{67b}$$

Consider the following functional:

$$\mathscr{F} = \mathscr{H} + a_1\mathscr{M} + a_2\mathscr{E} + a_3\mathscr{R}. \tag{68}$$

Computing the first variation of $\mathscr{F}$ gives

$$\delta\mathscr{F} = \int_{-\infty}^{+\infty} \left[\left(\frac{M}{\Gamma} + a_1 + a_3\epsilon \right)\delta M + \left(\frac{-\Gamma_\epsilon M^2}{2\Gamma^2} + a_2 + a_3 M \right)\delta\epsilon \right] dz. \tag{69}$$

We wish to find solutions such that $\delta\mathscr{F} = 0$. Clearly given any constant solution, we find a_1 and a_2 such that this condition is satisfied. Next we evaluate the second variation of $\mathscr{F}$ at the constant solution:

$$\delta^2\mathscr{F} = \int_{-\infty}^{+\infty} \left[\frac{(\delta M)^2}{\Gamma} + 2\left(a_3 - \frac{\Gamma_\epsilon M}{\Gamma^2} \right)\delta\epsilon\delta M + \left(\frac{2\Gamma_\epsilon^2 - \Gamma\Gamma_{\epsilon\epsilon}}{2\Gamma^4} \right)(\delta\epsilon)^2 \right] dz. \tag{70}$$

The definiteness of $\delta^2\mathscr{F}$ can be optimized by choosing a_3 as

$$a_3 = \frac{M\Gamma_\epsilon}{\Gamma^2}. \tag{71}$$

Notice that $-a_3$ is precisely the propagation velocity C of the energy density (see Whitman 1974), defined by

$$\frac{\partial H}{\partial t} + \frac{\partial CH}{\partial z} = 0. \tag{72}$$

Now given [71], it is apparent from [70] that the linear Lyapunov stability condition can be written as

$$\delta^2\mathscr{F} > 0 \quad \text{if } 2\Gamma_\epsilon^2 - \Gamma\Gamma_{\epsilon\epsilon} > 0. \tag{73}$$

Comparing [62] and [73] shows that the conditions for hyperbolicity and for linear Lyapunov stability coincide. In this problem, the higher order terms ($\delta^3\mathscr{F}$) are simple, and a straightforward argument could show that this is also the condition for nonlinear stability (see Holm *et al.* 1985).

Finally, we mention that if the hyperbolicity condition is met, then it is possible to show that the 2×2 system of equations given in [60] is genuinely nonlinear in the sense defined by Lax (1973). This means that the characteristic speeds increase along the eigenvectors. The upshot of this is that shock waves will develop in a finite time.

5. CONCLUSIONS

The present study exemplifies the complementarity between the variational approach based on an energy functional, and the averaging approach based on the local instantaneous equations. The former leads directly to the equations of motion, but does not keep track of the physical significance of each term appearing in the equations. On the other hand, the averaged equations help classify these terms into a set which is clearly identifiable. In this case:

- The "symmetric" virtual mass acceleration $\langle n_k \cdot n_z \Delta p'_{ki} \rangle_i$.
- The phase pressure difference term Δp_{ki}.
- The Reynolds stress due to the presence of the dispersed phase $\langle u_L'^2 \rangle$.

We first derive these terms by volume averaging. Then starting from the kinetic energy imparted to the continuous phase by the presence of the dispersed phase, we deduce the form of the symmetric virtual mass acceleration using Lagrangian variational methods.

In particular, it is shown that the virtual mass acceleration, the phasic pressure difference and the Reynolds stress associated with the velocity perturbation in the continuous phase, are three undissociable and complementary manifestations of the interactions between the phases in dispersed flows. Separate Bernoulli type equations are derived for the individual phases, and we identify an energy exchange term due to the relative velocity. Our analysis shows that the result from Voinov (1973) on the virtual mass was interpreted incorrectly, and that it actually leads to an objective acceleration in three dimensions. Furthermore, we show that in the dilute limit the model is never hyperbolic. It is conjectured that the effect of bubble interactions might make the model hyperbolic.

The Hamiltonian formalism of the model is investigated using a reduced set of equations based on the drift flux concept. It is shown that the linear Lyapunov stability condition coincides with the hyperbolicity condition.

Acknowledgements—P. Smereka was supported by Air Force Grants AFOSR-90-0090 and AFOSR-89-0165 at the Courant Institute and at the Institute for Mathematics and its Applications, respectively. This work was also supported, in part, by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation.

REFERENCES

AGEE, L. J., BANERJEE, S., DUFFEY, R. B. & HUGHES, E. D. 1978 Some aspects of two-fluid models and their numerical solution. *Proc. 2nd OECD Specialist Mtg on Transient Two-phase Flow, Paris*, **1**, 27–58.

BANERJEE, S. & CHAN, A. M. C. 1980 Separated flow model I. Analysis of the averaged and local instantaneous formulations. *Int. J. Multiphase Flow* **6**, 1–24.

BASSET, A. B. 1961 *A Treatise on Hydrodynamics*. Dover, New York.

BEDFORD, A. & DRUMHELLER, D. S. 1978 A variational theory of immiscible mixtures. *Archs Rat. Mech. Analysis* **68**, 37–51.

BENDIKSEN, K. H., ESPEDAL, M. & MALNES, D. 1988 Physical and numerical simulation of dynamic two-phase flow in pipe-lines with application to existing oil–gas filed lines. Presented at *Conf. on Multiphase Flow in Industrial Plants*, Bologna.

BIESHEUVEL, A. & VAN WIJNGAARDEN, L. 1984 Two-phase flow equations for a dilute dispersion of gas bubbles in liquid. *J. Fluid Mech.* **148**, 301–318.

BOURÉ, J. A. 1975 *On a Unified Presentation of the Non-equilibrium Two-phase Flows Models in Non-equilibrium Two-phase Flows* (Edited by LAHEY, R. T. JR & WALLIS, G. B.). *ASME Symposium Volume*. ASME, New York.

CLEBSCH, A. 1859 *J. Reine Angew. Math.* **56,** 1.

DELHAYE, J. M. & ACHARD, J. L. 1976 On the averaging operators introduced in two-phase flow modelling. *Proc. 1st OECD/NEA Specialist Mtg on Transient Two-phase Flow, Toronto,* **1,** 5–84.

DREW, D. A. 1983 Mathematical modelling of two-phase flow. *A. Rev. Fluid Mech.* **15,** 261–291.

DREW, D. A., CHENG, L. Y. & LAHEY, R. T. 1979 The analysis of virtual mass effects in two-phase flow. *Int. J. Multiphase Flow* **5,** 233–242.

DREW, D. A. & LAHEY, R. T. JR 1987 The virtual mass and lift force on a sphere in rotating and straining inviscid flow. *Int. J. Multiphase Flow* **13,** 113–121.

GEURST, J. A. 1985a Two-fluid hydrodynamics of bubbly liquid/vapour mixture including phase change. *Philips J. Res.* **40,** 352.

GEURST, J. A. 1985b Virtual mass in two-phase flow. *Physica* **129A,** 233–261.

GEURST, J. A. 1986 Variational principles and two-fluid hydrodynamics of bubbly liquid/gas mixtures. *Physica* **135A,** 455–486.

GURTIN, M. E. & STRUTHERS, A. 1991 Multiphase thermomechanics with interfacial structure 3. Evolving phase boundaries in the presence of bulk deformation. *Archs Rat. Mech. Analysis.* In press.

HOLM, D. D. & KUPERSHMIDT, B. A. 1986a Hamiltonian structure and Lyapunov stability of hyperbolic system of two-phase flow equations including surface tension. *Phys. Fluids* **29,** 989–991.

HOLM, D. D. & KUPERSHMIDT, B. A. 1986b A multipressure regularisation of multiphase flow. *Int. J. Multiphase Flow* **12,** 681–697.

HOLM, D. D. & KUPERSHMIDT, B. A. 1983 Poisson bracket and Clebsch representation for magnetohydrodynamics, multifluid plasmas, and elasticity. *Physica* **6D,** 347–363.

HOLM, D. D., MARSDEN, J. E., RATIU, T. & WEINSTEIN, A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123,** 1–115.

ISHII, M. 1975 *Thermofluid Dynamic Theory of Two-phase Flows.* Eyrolles, Paris.

LAX, P. D. 1973 Hyperbolic systems of conservation laws and the mathematical theory of shock waves. *SIAM Conf. Ser. on Applied Mathematics.*

LEECH, J. W. 1965 *Classical Mechanics.* Methuen, London.

LHUILLIER, D. 1985 Phenomenology of inertia effects in a dispersed solid–fluid mixture. *Int. J. Multiphase Flow* **11,** 427–444.

LIN, C. C. 1963 In *Liquid Helium. Proc. Enrico Fermi School of Physics, Course 21* (Edited by CARERI, G.). Academic Press, New York.

LIPTON, R. & AVELLANEDA, M. 1990 Darcy's law for slow viscous flow past a stationary array of bubbles. *Proc. R. Soc. Edinb.* **114A,** 71–79.

LYCKOWSKI, R. W. 1978 Theoretical basis of the drift-flux field equations and vapor drift velocity. *Proc. 6th Int. Heat Transfer Conf.*, Vol 1, pp. 339–344. Hemisphere, Washington, DC.

MICAELLI, J. C. 1987 CATHARE an advanced best estimate code for PWR safety analysis. Report SETh LEML-EM/87-58.

MILNE-THOMSON, L. M. 1968 *Theoretical Hydrodynamics,* 5th edn. Macmillan, New York.

MORISSON, P. J. 1981 Poisson brackets for fluids and plasmas. In *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems. AIP Conf. Proc., La Jolla* (Edited by TABOR, M. & TRÈVE, Y.), pp. 13–46.

NIGMATULIN, R. I. 1978 Averaging in mathematical modeling of heterogeneous and dispersed mixtures. Presented at the *International Center of Heat and Mass Transfer Symp.*, Yugoslavia.

NIGMATULIN, R. I. 1979 Spatial averaging in the mechanics of heterogeneous and dispersed systems. *Int J. Multiphase Flow* **5,** 353–385.

PANTON, R. J. 1978 Fluid properties for the continuum viewpoint of a non-equilibrium gas particle mixture. *J. Fluid Mech.* **31,** 273–303.

PAUCHON, C. 1989 A direct assessment of momentum interactions in incompressible two-phase flow. Presented at the *Int. Conf. on Mechanics of Two-phase Flows,* Taipei, Taiwan.

PAUCHON, C. & BANERJEE, S. 1986 Interphase momentum interaction effects in the averaged multifield model. Part I: void propagation in bubbly flows. *Int. J. Multiphase Flow* **12,** 559–573.

RANSOM, V. H., WAGNER, R. J. & TRAPP, J. A. 1981 The Relap5 two-phase fluid model and numerical scheme for economic LWR system simulation. *Proc. 3rd CSNI Specialist Mtg, Pasadena, CA,* pp. 513–531.

SELIGER, R. L. & WHITHAM, G. B. 1968 Variational principles in continuum mechanics. *Proc. R. Soc. Lond.* **A305,** 1.

SMEREKA, P. & MILTON, G. 1991 Bubbly flow and its relation to conduction in composites. *J. Fluid Mech.* **233,** 65–81.

TRAC-PD 2 1981 An advanced best estimate computer program for PWR loss of coolant accident analysis. Report LA-8709-MS.

VERNIER, P. & DELHAYE, J. M. 1968 General two-phase flow equations applied to the thermo-hydrodynamics of boiling water nuclear reactors. *Energie Primaire* **4,** 1–2.

VOINOV, D. V. 1973 On the force acting on a sphere in a non uniform flow of ideal incompressible fluid. *Zh. Prikl. Mekh. Tekh.* **4,** 182–184.

WALLIS, G. B. 1969 *One-dimensional Two-phase Flow.* McGraw-Hill, New York.

WALLIS, G. B. 1989 Inertial coupling in two-phase flows: macroscopic properties of suspensions in an inviscid fluid. *Multiphase Sci. Technol.* **5,** 239.

WALLIS, G. B. 1990 On Geurst's equations for inertial coupling in two-phase flow. In *Two Phase Flows and Waves* (Edited by JOSEPH, D. D. & SCHAEFFER, D. G.). *The IMA Volumes.* Springer, New York.

WALLIS, G. B. 1991 Interfacial pressure in inviscid two-phase dispersed flow. *Int. J. Multiphase Flow* **17,** 683–695.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley, New York.

YADIGAROGLU, G. & LAHEY, R. T. JR 1976 On the various forms of the conservation equations in two-phase flow. *Int. J. Multiphase Flow* **2,** 477–494.

ZUBER, N. 1964 On the dispersed two-phase flow in the laminar regime. *Chem. Engng Sci.* **19,** 897–917.

APPENDIX A

Here [14] and [15] are established. We will first show

$$\frac{1}{V}\int_{\bar{a}_1} p_{L1}\mathbf{n}_1\cdot\mathbf{n}_z\,dS=\frac{\partial}{\partial z}\left(\epsilon_G\frac{V}{a_1}\langle p_{L1}\rangle_1\right)+O\left(R\frac{\partial\epsilon_G}{\partial z}\right)+O\left(\frac{\epsilon_G^2 R}{H}\right). \tag{A.1}$$

To begin, we split this into two terms:

$$\frac{1}{V}\int_{a_1} p_{L1}\mathbf{n}_L\cdot\mathbf{n}_z\,dS=\frac{1}{V}\int_{\bar{a}_{1T}} p_{L1}\mathbf{n}_L\cdot\mathbf{n}_z\,dS+\frac{1}{V}\int_{\bar{a}_{1B}} p_{L1}\mathbf{n}_L\cdot\mathbf{n}_z\,dS, \tag{A.2}$$

where $\bar{a}_{1T}$ and $\bar{a}_{1B}$ are the surfaces of the bubble that intersect the averaging volume but are contained inside it. To proceed it is useful to introduce the following definition:

$$\bar{p}_{L1}(z')=\frac{1}{N_I 4\pi R^2}\sum_{j=1}^{N_I}\int_{B_j} p_{L1}\,dS, \tag{A.3}$$

where N_I is the number of spheres that intersect the plane z'; $\bar{p}_{L1}(z')$ is then the average interfacial pressure of the bubbles that intersect this plane. We now use [A.3] to rewrite [A.2] as

$$\begin{aligned}\frac{1}{V}\int_{\bar{a}_1} p_{L1}\mathbf{n}_L\cdot\mathbf{n}_z\,dS=&\frac{1}{V}\int_{\bar{a}_{1T}}[p_{L1}-\bar{p}_{L1}(z+H)]\mathbf{n}_L\cdot\mathbf{n}_z\,dS\\&+\frac{1}{V}\int_{\bar{a}_{1B}}[p_{L1}-\bar{p}_{L1}(z)]\mathbf{n}_L\cdot\mathbf{n}_z\,dS\\&+\frac{1}{V}\int_{\bar{a}_{1T}}\bar{p}_{L1}(z+H)\mathbf{n}_L\cdot\mathbf{n}_z\,dS\\&+\frac{1}{V}\int_{\bar{a}_{1B}}\bar{p}_{L1}(z)\mathbf{n}_L\cdot\mathbf{n}_z\,dS.\end{aligned} \tag{A.4}$$

By definition $\bar{p}_{\mathrm{L}i}$ does not vary when integrating around the bubbles at a fixed value of z. Therefore, the last two terms of [A.4] are equal to

$$\bar{p}_{\mathrm{L}i}(z+H)\frac{1}{V}\int_{\bar{a}_{i\mathrm{T}}}\mathbf{n}_{\mathrm{L}}\cdot\mathbf{n}_z\,\mathrm{d}S+\bar{p}_{\mathrm{L}i}(z)\frac{1}{V}\int_{\bar{a}_{i\mathrm{B}}}\mathbf{n}_{\mathrm{L}}\cdot\mathbf{n}_z\,\mathrm{d}S=\frac{1}{V}\{\bar{p}_{\mathrm{L}i}(z+H)A_{\mathrm{G}}(z+H)-\bar{p}_{\mathrm{L}i}(z)A_{\mathrm{G}}(z)\},$$

where $A_{\mathrm{G}}(z^*)$ is the area of the gas phase at the plane $z=z^*$. Now using the Leibnitz formula, the above expression is

$$=\frac{1}{V}\frac{\partial}{\partial z}\int_z^{z+H}\bar{p}_{\mathrm{L}i}(z')A_{\mathrm{G}}(z')\,\mathrm{d}z'. \qquad [A.5]$$

To study [A.5] further we split it up among the bubbles. Therefore, we write

$$\frac{1}{V}\int_z^{z+H}\bar{p}_{\mathrm{L}i}(z')A_{\mathrm{G}}(z')\,\mathrm{d}z'=\frac{1}{V}\sum_{j=1}^{N}\int_{z_j-R}^{z_j+R}\bar{p}_{\mathrm{L}i}(z')A_{\mathrm{G}j}(z')\,\mathrm{d}z', \qquad [A.6]$$

where $A_{\mathrm{G}j}(z^*)$ is the area of the jth bubble that intersects the plane $z=z^*$ and z_j is the location of the center of the jth bubble. Notice if $A_{\mathrm{G}j}(z^*)=0$ if z^* is outside the averaging volume and that

$$\int_{z_j-R}^{z_j+R}A_{\mathrm{G}j}(z')\,\mathrm{d}z'=\tau_j, \qquad [A.7]$$

where τ_j is the volume of the jth bubble that is contained in the averaging volume. Now we expand $\bar{p}_{\mathrm{L}i}(z')$ in a Taylor series about z_j. Therefore,

$$\bar{p}_{\mathrm{L}i}(z')=\bar{p}_{\mathrm{L}i}(z_j)+\bar{p}'_{\mathrm{L}i}(z_j)(z'-z_j)+\cdots. \qquad [A.8]$$

Clearly $\bar{p}'_{\mathrm{L}i}(z_j)$ must be small because if we change z_j by a small amount to z_j+h then the plane at $z=z_j+h$ must still intersect mostly the same bubbles. Indeed, using [A.3] one can establish that

$$\bar{p}'_{\mathrm{L}i}(z_k)=O\left(\frac{1}{\epsilon_{\mathrm{G}}}\frac{\partial\epsilon_{\mathrm{G}}}{\partial z}\right). \qquad [A.9]$$

This result explained simply by the fact that $\epsilon_{\mathrm{G}}^{-1}\,\partial_z\epsilon_{\mathrm{G}}$ represents the relative change in N_{I} as one varies z. The $\epsilon_{\mathrm{G}}^{-1}$ term seems to be problematic but we shall see below that it poses no problem. Substitution of [A.7]–[A.9] into [A.6] gives

$$\frac{1}{V}\int_z^{z+H}\bar{p}_{\mathrm{L}i}(z')A_{\mathrm{G}}(z')\,\mathrm{d}z=\frac{1}{V}\sum_{j=1}^{N}\bar{p}_{\mathrm{L}i}(z_j)\tau_j+O\left(R\frac{\partial\epsilon_{\mathrm{G}}}{\partial z}\right). \qquad [A.10]$$

We write the sum as

$$\frac{1}{V}\sum_{j=1}^{N}\bar{p}_{\mathrm{L}i}(z)\tau_j=\frac{1}{V}\tfrac{4}{3}\pi R^3\sum_{j=1}^{\tilde{N}}\bar{p}_{\mathrm{L}i}(z_j)+\frac{1}{V}\sum_{j=1}^{\bar{N}}\bar{p}_{\mathrm{L}i}(z_j)\tau_j, \qquad [A.11]$$

where $\tilde{N}$ is the number of bubbles that do not intersect the averaging volume and $\bar{N}$ is the number of bubbles that do. Since $\bar{N}$ is $O(\epsilon_{\mathrm{G}}RN/H)$ then it follows that the second term in [A.11] is $O(\epsilon_{\mathrm{G}}R/H)$. Further, it follows from the definition of $\bar{p}_{\mathrm{L}i}$ that the first term of [A.11] is equal to

$$\frac{4\pi R^3}{3V}\sum_{j=1}^{\tilde{N}}\frac{1}{4\pi R^2}\int_{B_j}p_{\mathrm{L}i}\,\mathrm{d}S. \qquad [A.12]$$

Therefore, combining [A.12], [A.11] and [A.10], we have:

$$\frac{1}{V}\int_z^{z+H}\bar{p}_{\mathrm{L}i}(z')A_{\mathrm{G}}(z')\,\mathrm{d}z'=\frac{4\pi R^3}{3V}\sum_{j=1}^{\tilde{N}}\int_{B_j}p_{\mathrm{L}i}\,\mathrm{d}S+O\left(R\frac{\partial\epsilon_{\mathrm{G}}}{\partial z}\right)+O\left(\epsilon_{\mathrm{G}}^2\frac{R}{H}\right);$$

which to the same approximation is

$$\frac{1}{V}\int_z^{z+H}\bar{p}_{\mathrm{L}i}(z')A_{\mathrm{G}}(z')\,\mathrm{d}z=\frac{V}{a_i}\epsilon_{\mathrm{G}}\langle p_{\mathrm{L}i}\rangle+O\left(R\frac{\partial\epsilon_{\mathrm{G}}}{\partial z}\right)+O\left(\frac{\epsilon_{\mathrm{G}}^2}{H}R\right). \qquad [A.13]$$

Now substituting [A.13] into [A.5] and combining with [A.4], we have:

$$\frac{1}{V}\int_{\bar{a}_i} p_{Li}\mathbf{n}_L\cdot\mathbf{n}_z\,dS=\frac{\partial}{\partial z}\left(\epsilon_G\frac{V}{a_i}\langle p_{Li}\rangle_i\right)+O\left(R\frac{\partial\epsilon_G}{\partial z}\right)+O\left(\frac{\epsilon_G^2}{H}R\right)$$
$$+\frac{1}{V}\int_{\bar{a}_{iT}}[p_{Li}-\bar{p}_i(z+H)]\mathbf{n}_L\cdot\mathbf{n}_z\,dS+\frac{1}{V}\int_{\bar{a}_{iB}}[p_{Li}-\bar{p}_i(z)]\mathbf{n}_L\cdot\mathbf{n}_z\,dS.$$

The last two terms must be small since the average part has been subtracted out. Therefore we conclude thay they must be $O(\tau\bar{N}/V)=O(\epsilon_G^2R/H)$. Thus, [A.1] has been established.

Next we turn to demonstrating that [15] is true. We have

$$\frac{1}{V}\int_{\bar{a}_i} p_{Li}\mathbf{n}_L\cdot\mathbf{n}_z\,dS=\frac{1}{V}\sum_{j=1}^{\tilde{N}}\int_{B_j} p_{Li}\mathbf{n}_L\cdot\mathbf{n}_z\,dS, \qquad [A.14]$$

where B_j is the jth bubble totally enclosed by the averaging volume and $\tilde{N}$ is the number of such bubbles. To develop this expression further we write it as

$$\frac{1}{V}\sum_{j=1}^{\tilde{N}}\int_{B_j} p_{Li}\mathbf{n}_L\cdot\mathbf{n}_z\,dS=-\frac{1}{V}\sum_{j=1}^{\tilde{N}}\int_{B_j}(p_{Li}-p_{j0})\mathbf{n}_G\cdot\mathbf{n}_z\,dS-\frac{1}{V}\sum_{j=1}^{\tilde{N}}\int_{B_j} p_{j0}\mathbf{n}_G\cdot\mathbf{n}_z\,dS, \qquad [A.15]$$

where p_{j0} represents the pressure field if the jth bubble was not present. The reader is reminded that here we have used $\mathbf{n}_L=-\mathbf{n}_G$. Since p_{j0} is the pressure field when there is no bubble present it must be close to the average liquid pressure. However, it will vary slightly within the averaging volume. We expand p_{j0} about the average liquid pressure:

$$p_{j0}=\langle p_L\rangle+\frac{\partial\langle p_L\rangle}{\partial z}(z'-z_{mid})+\cdots, \qquad [A.16]$$

where z' is the coordinate in the averaging volume and z_{mid} is the middle of the averaging volume. Substituting this into [A.15] gives:

$$\frac{1}{V}\sum_{j=1}^{\tilde{N}}\int_{B_j} p_{j0}\mathbf{n}_G\cdot\mathbf{n}_z\,dS=\frac{\tilde{N}\tau}{V}\frac{\partial\langle p_L\rangle}{\partial z}+\text{h.o.t.}$$
$$=\epsilon_G\frac{\partial\langle p_L\rangle}{\partial z}+\text{h.o.t.}, \qquad [A.17]$$

since

$$\epsilon_G=\frac{\tilde{N}\tau}{V}+O\left(\frac{\epsilon_G^2R}{H}\right); \qquad [A.18]$$

where H is the height of the averaging volume. The second term in [A.18] arises from the fact the bubbles on the boundary make a very small contribution to the void fraction. $p_{Li}-p_{j0}$ represents the disturbance in the pressure field caused by the introduction of a bubble. Since we have assumed that all bubbles in the averaging volume have the same velocity then it follows that

$$-\frac{1}{V}\sum_{j=1}^{\tilde{N}}\int_{B_j}(p_{Li}-p_{j0})\mathbf{n}_G\cdot\mathbf{n}_z\,dS=-\frac{\epsilon_G}{\frac{4}{3}\pi R^3}\int_B p'_{Li}\mathbf{n}_G\cdot\mathbf{n}_z\,dS+O\left(\frac{\epsilon_G^2R}{H}\right), \qquad [A.19]$$

where $p'_{Li}=p_{Li}-p_{L0}$ and we have used [A.18]. The higher order term is similar to the previous equation. Substituting [A.17] and [A.19] into [A.15] gives [15].

APPENDIX B

Here we derive several formulas needed in section 2.2. In order for the averaged equations to be Galilean invariant it is important to include gradients in the calculation of these terms. We are interested in potential flow around a sphere, but first we consider the velocity potential without the presence of the sphere. Since the flow is incompressible we have

$$\mathbf{u}_L=-\nabla\Phi_0,\quad\text{where }\nabla^2\phi_0=0; \qquad [B.1]$$

where ϕ_0 is the undisturbed velocity potential. The simplest potential function that produces a liquid velocity with a gradient is

$$\phi_0 = -v(t)r\cos\theta - \frac{a(t)}{4}r^2(3\cos^2\theta - 1), \qquad [B.2]$$

where r and θ are spherical coordinates and $z = r\cos\theta$. From [B.2] it is easy to see that

$$\mathbf{u}_L \cdot \mathbf{n}_z = v(t) + a(t)z \qquad [B.3]$$

and

$$u_L(z=0) = v(t) \quad \text{and} \quad \left.\frac{\partial u_L}{\partial z}\right|_{z=0} = a(t). \qquad [B.4]$$

Now we introduce a sphere of radius R into the flow that is placed at $z = 0$ and is moving with a velocity $u_G = u_G(z, t)$ in the z-direction. We look for a new velocity potential satisfying:

(i) $\phi(r \to \infty) = \phi_0$.
(ii) $\nabla^2\phi = 0$.
(iii) $\nabla\phi \cdot \mathbf{n}_G = -u_G(\mathbf{n}_z \cdot \mathbf{n}_G)$.

The solution to this problem can be computed using Weiss's sphere theorem (see Milne-Thomson 1968) and the result is

$$\phi = \phi_0 + \phi',$$

where

$$\phi' = \frac{(u_G - v)R^3\cos\theta}{2r^2} - \frac{aR^5}{6r^3}(3\cos^2\theta - 1), \qquad [B.5]$$

and we are now in the frame moving with the bubble. The pressure equation in a frame moving with the bubble is

$$\frac{p_L}{\rho} + \tfrac{1}{2}|\nabla\phi|^2 - \left(\frac{\partial\phi}{\partial t}\right)_G + u_G\mathbf{n}_z \cdot \nabla\phi = c(t),$$

where the subscript G emphasizes that the time derivative is really a material derivative. Since $p_L(r \to \infty) = p_0(r \to \infty)$, where p_0 is the pressure with no bubble we can eliminate $c(t)$ and we find

$$\frac{p_L - p_0}{\rho_L} = \left(\frac{\partial\phi'}{\partial t}\right)_G - \tfrac{1}{2}|\nabla\phi'|^2 - (u_G\mathbf{n}_z + \nabla\phi_0)\cdot\nabla\phi'. \qquad [B.6]$$

Therefore, we have

$$\int_B \frac{p_L'}{\rho_L}\mathbf{n}_G \cdot \mathbf{n}_z \, dS = 2\pi R^2 \int_0^\pi \left[\left(\frac{\partial\phi'}{\partial t}\right)_G - \tfrac{1}{2}|\nabla\phi'|^2 - (u_G\mathbf{n}_z + \nabla\phi_0)\cdot\nabla\phi'\right]\sin\theta\cos\theta \; d\theta. \qquad [B.7]$$

Substitution of ϕ_0 and ϕ' into [B.7] yields, after straightforward calculations,

$$\int_B p_L'\mathbf{n}_G \cdot \mathbf{n}_z \, dS = \tfrac{2}{3}\pi R^3\rho_L\left\{\left(\frac{\partial u_G}{\partial t}\right)_G - \frac{\partial v}{\partial t} + a(t)[u_G(t) - v(t)]\right\}. \qquad [B.8]$$

Substituting [B.4] into [B.8], we have

$$\int_B p_L'\mathbf{n}_G \cdot \mathbf{n}_z \, dS = \tfrac{2}{3}\pi R^3\rho_L\left[\left(\frac{\partial u_G}{\partial t}\right)_G - \left(\frac{\partial u_L}{\partial t}\right)_G + (u_G - u_L)\frac{\partial u_L}{\partial z}\right]_{z=0}.$$

Transforming back to the lab frame we have

$$\left(\frac{\partial u_G}{\partial t}\right)_G = \frac{\partial u_G}{\partial t} + u_G\frac{\partial u_G}{\partial z}$$

and

$$\left(\frac{\partial u_L}{\partial t}\right)_G = \frac{\partial u_L}{\partial t} + u_G\frac{\partial u_L}{\partial z}.$$

Interpreting u_L as an average velocity we have

$$\int_B p'_L \mathbf{n}_G \cdot \mathbf{n}_z \, dS = \tfrac{2}{3}\pi R^3 \rho_L \left(\frac{D_G u_G}{Dt} - \frac{D_L \langle u_L \rangle}{Dt} \right) + O(R^5). \qquad [B.9]$$

However, we have ignored higher order effects by our choice of potential function. We infer from the work of Voinov (1973) that the error in the above expression is $O(R^5)$. The velocity fluctuation term is

$$\int_{V_L} u_L'^2 \, dV = \int_{V_L} |\nabla(\phi - \phi_0)|^2 \, dV$$

$$= \frac{2\pi R^3}{3} (u_G - v)^2 + O(R^5). \qquad [B.10]$$

Also we, have

$$\int_B (p_L - \langle p_L \rangle) \, dS = \int_B (p_L - p_0) \, dS + \int_B (p_0 - \langle p_L \rangle) \, dS.$$

The first term is found by using [B.6] and the second term using [17]. The second term is zero except for $O(R^5)$ terms. We find

$$\int_B (p_L - \langle p_L \rangle) \, dS = -\rho_L \pi R^2 (u_G - v)^2 + O(R^5). \qquad [B.11]$$

Both [B.10] and [B.11] are revealed by straightforward integration in spherical coordinates.